Geometric entropy, area, and strong subadditivity

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Abstract

The trace over the degrees of freedom located in a subset of the space transforms the vacuum state into a mixed density matrix with non zero entropy. This geometric entropy is believed to be deeply related to the entropy of black holes. Indeed, previous calculations in the context of quantum field theory, where the result is actually ultraviolet divergent, have shown that the geometric entropy is proportional to the area for a very special type of subsets. In this work we show that the area law follows in general from simple considerations based on quantum mechanics and relativity. An essential ingredient in our approach is the strong subadditive property of the quantum mechanical entropy.

I. INTRODUCTION

A chain of arguments using thermodynamics, classical general relativity and quantum field theory in curved spaces shows that the black hole has an associated entropy proportional to the horizon area. This is arguably one of the most important clues to the understanding of the role of gravity at the quantum level. However, these calculations give no precise identification of the degrees of freedom responsible for the entropy, nor an explanation of its relation with area. There are several approaches in this regard, involving either specific models of quantum gravity or effective ideas [1].

A very appealing candidate in this last sense is given by the entropy of entanglement [2,3]. On the black hole spacetime we can write the Hilbert space of a quantum field as a tensor product of two Hilbert spaces, which correspond to the asymptotic data at infinity and at the horizon respectively. The quantum state relevant for asymptotic observers is obtained by tracing the complete quantum state density matrix $|\Psi\rangle\langle\Psi|$ over the invisible degrees of freedom on the horizon. But $|\Psi\rangle$ contains correlations between the field modes inside and outside the black hole, or, in other words, it is entangled with respect to the above tensor product. Thus, the partial trace leads to a mixed density matrix with non zero entropy.

For large black holes the near horizon geometry is approximately flat. This suggests that the relevant entropy should also occur in flat space [4]. To mimic the black hole in Minkowski space take a spatial hyperplane and a subset X on it. This divides the space in two, the interior and exterior regions with respect to X. The trace of the vacuum state over

the interior (or exterior) degrees of freedom is a mixed density matrix. Its entropy, being a function of the subset X, was called geometric entropy [4]. This new flat space quantity, suggested by black hole physics, is interesting in its own right, and provides a simpler context to contrast ideas.

Several authors have done explicit calculations of the geometric entropy in quantum field theory (QFT). The methods ranged from numerical work [3,5], to analytical calculations based on the replica trick, the heat kernel, and conformal symmetry in two dimensional spacetime [4,6–10]. In all cases, the reassuring result is an entropy proportional to the area. However, most of the work has focused on free fields and a very restricted class of subsets. The application of the analytical methods was constrained to the infinite half space since they use in an essential way the fact that density matrix corresponding to the Rindler wedge is a thermal one [10]. The numerical methods encounter practical problems for non spherical subsets (see however the recent work on annular and flower-like geometries [11,12]). Also, the physical principles behind the proportionality between area and entropy remain obscure beyond the explicit calculations, that involve dealing with non renormalizable divergences.

In this work we show that for a wide class of subsets the area law for the geometric entropy follows from very general and model independent ideas based on quantum mechanics and relativity.

We assume, as it has been done explicitly or implicitly in all the previous work on the subject, that it is possible to divide a Cauchy surface into pieces and trace the vacuum state over the degrees of freedom in one of them to obtain a density matrix. This is the single basic condition allowing to define a geometric entropy. However, this is a delicate point, related to the divergences appearing in the QFT calculations. We will only say a bit more about it in Section III.

The causal evolution is unitary. Accordingly, the geometric entropy corresponding to two space-like subsets coincide if they give place to the same causal domain of dependence, or, what is the same, they can be extended to a global Cauchy surface by union with the same space-like subset. We also make use of the Poincare symmetry of the vacuum state.

It is essential to the arguments below the mathematical fact that the quantum mechanical entropy is positive and strongly subadditive. The strong subadditive property was introduced in [13,14], in the context of the investigations around an old problem in statistical mechanics, namely, the proof of the existence of the mean entropy (the limit of entropy over volume for large sets). Here we apply these ideas to the relativistic situation. Note that for a general quantum statistical mechanical system the entropy in a given volume has to be defined in the same way as the geometric entropy is defined above, that is, by tracing over degrees of freedom in the appropriate domain. In this sense, the geometric entropy is just the name for the more familiar concept of the entropy contained in a volume in the special case where the state is the relativistic vacuum.

The cited investigations on the Euclidean symmetric situation where mainly focused on the extensive properties of the entropy on large domains. Here the mean entropy is zero. Basically, this is so because we can construct Cauchy surfaces with arbitrarily small volume by approaching the null surfaces, and use the boost symmetries and the subadittive property to put an upper bound to the entropy which grows like the area. Moreover, the strong subadditivity allows to completely determine the functional form of the geometric entropy. Curiously, a very relevant piece of the information it provides, that gives the clue

to show that the geometric entropy is not only bounded by the area but linearly growing with it, shows up already when combined only with translation invariance. However, the crucial limit for the geometric entropy is not the one of infinite volume but that of the very flat type of sets.

Summarizing, we show that the combination of strong subadditivity, Poincare symmetry and causality constrains the geometric entropy to be proportional to the bounding area, plus a constant term.

The work is organized as follows. In Section II we review some properties of the quantum entropy. In Section III we state the problem of the geometric entropy in Euclidean and relativistic spaces. In Section IV we review some work on the entropy for Euclidean symmetric states and develop some new results on this subject, related to the entropy of very flat sets. The main result of the paper is the theorem in Section V which states that geometric entropy is proportional to the area. Finally we present the conclusions and discuss some open perspectives.

II. PROPERTIES OF THE QUANTUM ENTROPY

In this Section we review some inequalities satisfied by the quantum entropy. The proofs of the statements, the original references and further details can be found in the reviews [15,16].

In quantum mechanics a physical state is described by a density matrix ρ, which is a self-adjoint and positive operator with unit trace, $Tr\rho = 1$, defined on a Hilbert space $\mathcal{H}$. The entropy S is a function of the quantum state given by the expression

$$S = -Tr\rho \log \rho \,. \tag{1}$$

The quantum entropy is always positive, $S \geq 0$, being zero if and only if the state is pure, that is, the density matrix is a one dimensional projector $\rho = |\Psi\rangle\langle\Psi|$. The entropy can be infinite for infinite dimensional Hilbert spaces, while on a space with finite dimension d it is bounded above by $\log d$. The function S satisfies

$$\lambda_1 S(\rho_1) + \lambda_2 S(\rho_2) - \lambda_1 \log(\lambda_1) - \lambda_2 \log(\lambda_2) \geq S(\lambda_1\rho_1 + \lambda_2\rho_2) \geq \lambda_1 S(\rho_1) + \lambda_2 S(\rho_2)\,, \tag{2}$$

where λ_1 and λ_2 are any positive numbers such that $\lambda_1 + \lambda_2 = 1$. The second inequality means that the effect of mixing always increases the entropy. Moreover, any state can be decomposed as a mixing of pure states.

In the following we will be mainly interested in the properties of the entropy related to spaces constructed out of the tensor product of two or more Hilbert spaces. Let the Hilbert space $\mathcal{H} = \mathcal{H}_1 \bigotimes \mathcal{H}_2$ and $\rho = \rho_{i_1 i_2, i'_1 i'_2}$ be the density matrix on $\mathcal{H}$. By tracing over $\mathcal{H}_1$ we can form the density matrix $\rho_2 \equiv (\rho_2)_{i_2 i'_2} = \sum_{i_1} \rho_{i_1 i_2, i_1 i'_2}$ on $\mathcal{H}_2$. The density matrix ρ_1 is defined similarly by tracing over $\mathcal{H}_2$.

For a pure density matrix ρ the following duality property for the entropy of the subsystems holds

$$S(\rho_1) = S(\rho_2)\,. \tag{3}$$

This follows from the fact that for $\rho = |\Psi\rangle\langle\Psi|$ we have the equality $Tr(\rho_1^n) = Tr(\rho_2^n)$ for any power n, and consequently the non zero eigenvalues and their multiplicities coincide for ρ_1 and ρ_2.

It is easy to give examples where ρ is pure, so that $S(\rho) = 0$, while ρ_1 is in a mixed state with $S(\rho_1) > 0$. Thus, the quantum entropy can not be generically increasing (monotonicity) with the size (the number of degrees of freedom) of the subsystem (nor with the size of the subsystem that is traced over). This is somehow against the intuition coming from the extensive entropy of gases.

A natural question is what type of density matrices ρ_1 can arise from tracing a pure density matrix ρ over a subsystem. The answer is that any density matrix can be obtained in this way. A density matrix ρ_1 on a space $\mathcal{H}$ can always be realized as the partial trace of a pure density matrix ρ on $\mathcal{H} \bigotimes \mathcal{H}$.

The entropy for a system composed by two independent subsystems is the sum of the subsystem entropies. This is at the physical basis of the explicit form of S given by equation (1). The density matrix for the composed system is given by the tensor product $\rho = \rho_1 \bigotimes \rho_2$ in this case, and we have the equation

$$S(\rho) = S(\rho_1) + S(\rho_2)\,. \tag{4}$$

In the general case where ρ does not factorize and the subsystems can not be considered as statistically independent, this equation can be extended to an inequality expressing the subadditive property of the entropy

$$S(\rho) \leq S(\rho_1) + S(\rho_2)\,. \tag{5}$$

In fact the entropy satisfies a stronger inequality. To introduce it, it is convenient to consider a more general case in which the Hilbert space of the system is a tensor product of an arbitrary number of factors $\mathcal{H} = \bigotimes_{i\in I} \mathcal{H}_i$, where I is the set of indices labeling the different subsystem Hilbert spaces $\mathcal{H}_i$. Let A be any subset of the set of indices I. Define the corresponding reduced density matrix on $\bigotimes_{i\in A} \mathcal{H}_i$, by tracing over all $\mathcal{H}_i$ with $i \notin A$,

$$\rho_A = Tr_{\bigotimes_{i\in -A} \mathcal{H}_i} \rho\,. \tag{6}$$

Call $S(A) \equiv S(\rho_A)$ the corresponding entropy. With this notation the subadditive property (5) writes

$$S(A) + S(B) \geq S(A \cup B) \tag{7}$$

for any pair of subsets A and B of I, while the duality relation for a pure total density matrix ρ is simply

$$S(A) = S(-A)\,. \tag{8}$$

Here $-A$ is the set complementary to A in I. As mentioned, I can be artificially enlarged to a bigger set I', and the total density matrix can be taken pure in the enlarged space, without modifying the value of $S(A)$ for $A \subseteq I$. In this case (8) holds in the bigger space I'.

Applying subadditivity to the complements $-A$ and $-B$ we have $S(-A) + S(-B) \geq S(-(A \cap B))$, since the complement operation interchanges union and intersection. Thus, purifying the total density matrix the inequality (5) implies a different relation, namely

$$S(A)+S(B)\geq S(A\cap B). \tag{9}$$

The strong subadditivity (SSA) generalizes the relations (7) and (9) in a form that is self-dual under taking complements in the pure case. It writes

$$S(A)+S(B)\geq S(A\cup B)+S(A\cap B)\,. \tag{10}$$

Using the trick of working with an artificially purified density matrix the SSA applied to A and $-B$ leads to a different self-dual relation. This is

$$S(A)+S(B)\geq S(A-B)+S(B-A)\,, \tag{11}$$

where $A-B$ means $A\cap(-B)$. The inequalities (10) and (11) are equivalent in the pure case but they are both valid in general. In lack of a better name, and since it is closely related to SSA, we often call SSB the inequality (11).

Finally, we also mention that for two non intersecting sets A and B the SSA and SSB relations lead to the following triangle inequalities for the entropies

$$|S(A)-S(B)|\leq S(A\cup B)\leq S(A)+S(B)\,. \tag{12}$$

The second one is of course subadditivity. The inequalities (12) are useful to prove some continuity properties of S. For example, if the entropy of B is very small one has that $S(A)$ and $S(A\cup B)$ are very near to each other.

We mentioned that S is not monotonic in general. By monotonicity here we mean the property $S(B)\leq S(A)$ for any $B\subseteq A$. However, the inequality (11), or the first one from (12), can be thought as partial compensations for the lack of monotonicity.

Inequalities provide less concrete information and are more difficult to handle than equations. This is perhaps the reason why the relations (10) and (11) are relatively poorly known and have found only few applications. However, in the presence of symmetries these can be very useful as will become apparent in the following.

III. STRONG SUBADDITIVITY AND CONTINUUM QUANTUM SYSTEMS

Let us consider a continuous quantum system in $\mathbf{R}^d$ described in terms of a Fock space. The single particle states are given by elements of $\mathcal{H}_1(\mathbf{R}^d)=\mathcal{L}^2(\mathbf{R}^d)$, the Hilbert space of square integrable functions in $\mathbf{R}^d$, while the n particle space is the product of n copies of $\mathcal{H}_1(\mathbf{R}^d)$, $\mathcal{H}_n(\mathbf{R}^d)=\mathcal{H}_1(\mathbf{R}^d)\otimes\ldots\otimes\mathcal{H}_1(\mathbf{R}^d)$, where the tensor product is understood as symmetrized or antisymmetrized according to the particles being bosons or fermions. The Fock space is then given by the direct sum $\mathcal{H}=\bigoplus_0^\infty\mathcal{H}_n$. Similarly, we can define the single particle states in a given volume V in $\mathbf{R}^d$ as $\mathcal{H}_1(V)=\mathcal{L}^2(V)$, and its corresponding Fock space $\mathcal{H}(V)=\bigoplus_0^\infty\mathcal{H}_n(V)$. If the intersection of two sets V_1 and V_2 has zero measure we have

$$\mathcal{H}(V_1\cup V_2)=\mathcal{H}(V_1)\bigotimes\mathcal{H}(V_2)\,. \tag{13}$$

Now, let the density matrix of the system be ρ. We can define a density matrix ρ_V for a volume V by tracing ρ over $\mathcal{H}(-V)$. Its entropy is $S(V)=-Tr\rho_V\log(\rho_V)$. Thus, the density matrices corresponding to the subsets satisfy the compatibility condition

$$\rho_V = Tr_{\mathcal{H}(V')} \rho_{V\cup V'} \, , \tag{14}$$

where V' is any set that has measure zero intersection with V. A continuous quantum system is defined here as a set of Hilbert spaces and density matrices corresponding to subsets of $\mathbf{R}^d$ satisfying the conditions (13) and (14). The strong subadditive inequalities then take a geometric form given by (10) and (11), where now A and B represent subsets of $\mathbf{R}^d$ [15].

The problem of building a continuous quantum system can be presented in a slightly different way where the complete system is constructed out of the finite volume subsystems. These may be defined for example by giving a Hamiltonian and boundary conditions, and taking a finite volume density matrix ρ_V. Then, to glue all elements into a unique system we have to impose (13) and (14) on the local Hilbert spaces and states.

When the total state ρ is invariant under some transformation $\mathcal{U}$ in $\mathbf{R}^d$ the entropy $S(\mathcal{U}(A))$ is equal to $S(A)$. This symmetry combined with the properties of the entropy can strongly constrain the function S. We say that a positive function S on subsets of $\mathbf{R}^d$ is an Euclidean entropy function if it is invariant under rotations and translations, and satisfies SSA and SSB.

Relativistic setting

In the relativistic case a pair of time-like related subsets contains degrees of freedom that are not mutually independent. A scheme as the previous one for an Euclidean system, where (13) and (14) hold, must correspond to every Cauchy surface in spacetime.

All previous work on geometric entropy assumes that we can divide a Cauchy surface for the spacetime into pieces and trace the total state over the degrees of freedom on one of them to obtain a density matrix. This assumption is also made here, since it is fundamental to the very definition of a geometric entropy.

However, this is at the root of the divergences encountered in quantum field theory calculations. The complete localization of states in a given region is forbidden by standard axioms in QFT (see [17] for a clear discussion on this point). The covariant regularizations that deal with the infinites arising in perturbation theory can not in general avoid the divergence in the entropy, which shows up at the level of free fields. Also, this is not softened by supersymmetry. Thus, as these divergences suggest, there are two types of enigma related to the black hole entropy formula. The first is given by the proportionality between area and entropy, and the second one is that the entropy is surprisingly finite. Here we do not deal with this last and probably deeper problem. It is possible that a finite entropy would require quantum gravity input (see also a different idea in [18]). However, the existence of a good covariant definition for a finite geometric entropy in the context of QFT can not be discarded. But, in that case, the role of the set boundaries can not be taken as purely passive. In other words, to define a bounded system Hilbert space, boundary conditions are needed. The precise compatibility conditions that these bounded systems should have in order to be used for computing a geometric entropy are given by (13) and (14). This will be enough characterization for our purposes. Any non ambiguous definition for a localized entropy beyond the classical limit should suffer from the same problem. In particular, this is the case for the various entropy bounds present in the literature, which are suggested by ideas related to the physics of black holes [19]. In the semiclassical level they should take

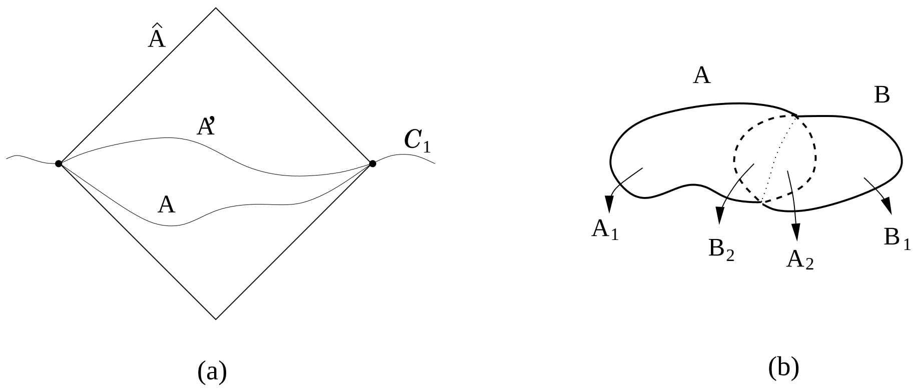

FIG. 1. (a)- A causally closed set $\hat{A}$ in $1+1$ dimensions. It is the causal development of the spatial surfaces A and A'. We say that A and A' are Cauchy surfaces for $\hat{A}$. Both of these sets can be continued to a Cauchy surface for the Minkowski space using the same spatial set $\mathcal{C}_1$. The marked points on the left and right corners of the diamond shaped set $\hat{A}$ represent the spatial corner of $\hat{A}$. (b)- Two commuting causally closed sets in $2+1$ represented here by their Cauchy surfaces representatives A and B. These surfaces intersect in the dotted line in this example, dividing A into A_1 and A_2 and B into B_1 and B_2. The commutation imposes that $A_2 \subseteq \hat{B}$ and $B_2 \subseteq \hat{A}$. Note that the spatial corners of $\hat{A}$ and $\hat{B}$, which are the boundaries of A and B respectively, are spatial to each other. The causally closed sets $\hat{A} \vee \hat{B}$ and $\hat{A} \wedge \hat{B}$ that enter into the SSA relation are generated by $A_1 \cup B_1$ and $A_2 \cup B_2$ respectively.

into account the geometric entropy of quantum origin, what can make them much more constraining and interesting [20].

Therefore, here we assign a density matrix and an entropy to any subset A of a Cauchy surface $\mathcal{C}$. In the following we restrict the analysis to the A with compact closure. The conditions (13) and (14) are assumed to hold among the subsets of any such surface $\mathcal{C}$.

The causal development $\hat{A}$ of A is the set where the solution of a wave like equation is uniquely determined by the initial data on A. More generally, for any set O not necessarily space-like, the causal development is the set of points x such that every inextensible time-like curve through x cuts O. We often say, abusing of the terminology, that a space-like set A is a Cauchy surface for $\hat{A}$. Consider two subsets A and A' of different Cauchy surfaces $\mathcal{C}$ and $\mathcal{C}'$, but having the same causal development $\hat{A} = \hat{A}'$ (see fig.(1)a). We can extend A and A' with the same set to form a global Cauchy surface, that is, we can choose $\mathcal{C}_1 = \mathcal{C} - A = \mathcal{C}' - A'$. The unitarity of the causal evolution implies that we have

$$S(A) = S(A') \equiv S(\hat{A})\,. \tag{15}$$

This means that there is no loss nor gain of information in passing from A to A'. In the Heisenberg picture both $\rho(A)$ and $\rho(A')$ are obtained by tracing the total state over the same degrees of freedom in $\mathcal{C}_1 = \mathcal{C} - A = \mathcal{C}' - A'$.

Then, $\hat{A}$ can be seen as an equivalence class of pieces of Cauchy surfaces with the same causal development, density matrix and entropy. Note that all Cauchy surfaces for $\hat{A}$ share the same boundary, the spatial corner of $\hat{A}$ (the subset of the closure of $\hat{A}$ that is not time-like related to any point in A, see the figure (1)). The sets of the type $\hat{A}$ are called here causally closed, and they constitute the natural domain of S in the relativistic case. We

often make no distinction between the set $\hat{A}$ and an element A on the equivalence class it represents, and denote them with the same capital letter.

Now we identify the conditions allowing to apply the strong subadditive property for any two given causally closed sets $\hat{A}$ and $\hat{B}$. As a consequence of eqs. (13) and (14) the SSA and SSB inequalities in the form (10) and (11) take place for the entropies of two subsets A and B of the same global Cauchy surface $\mathcal{C}$. In that case the sets $A \cap B$ and $A \cup B$ appearing in the smaller side of the strong subadditive relation are Cauchy surfaces for $\hat{A} \cap \hat{B}$ and the the causal development of $\hat{A} \cup \hat{B}$ respectively. However, it is possible to apply SSA and SSB even in some cases where A and B do not belong to the same $\mathcal{C}$. As the entropies really depend on the equivalence classes $\hat{A}$ and $\hat{B}$, this happens if there are different Cauchy surfaces A' and B' for $\hat{A}$ and $\hat{B}$ that belong to the same global Cauchy surface $\mathcal{C}'$. We say that two causally closed sets $\hat{A}$ and $\hat{B}$ (or any pair of Cauchy surfaces for them) commute if there is at least a pair of the respective Cauchy surface representatives that belong to the same global Cauchy surface (see fig.(1)b). It can be seen that an equivalent condition for the commutativity of $\hat{A}$ and $\hat{B}$ is that their spatial corners are spatial to each other. In particular, if $\hat{A}$ includes $\hat{B}$ or they are spatially separated, they commute. The inequalities SSA or SSB can be applied only for the entropies of commuting elements. For these, the right hand side of (10) and (11) has to be calculated with any pair of Cauchy surface representatives belonging to the same global Cauchy surface.

There is a more elegant way to define the concept of causally closed sets and commutativity that has interesting connections with quantum mechanics [21]. Write $x \sim y$ for two points that are time-like related. Given any subset O of the spacetime define its causal opposite $O^{\perp}$ as the subset of points which are spatial to O, $O^{\perp} = \{x / x \not\sim y, x \neq y, \text{for every}\, y \in O\}$. The causally closed sets are just the subsets O which coincide with their double opposite, $O^{\perp\perp} = O$ [1]. Curiously, this sets form an orthomodular lattice (also called quantum logic). This structure is also shared by the physical propositions in quantum mechanics, or equivalently, the orthogonal projectors on the Hilbert space. In any orthomodular lattice there are three operations, the opposite, the meet $A \wedge B$ and the join $A \vee B$ of two elements, and an order relation $\subseteq$. In the present context they are given respectively by the above opposite, the intersection $A \cap B$, the double opposite of the union $(A \cup B)^{\perp\perp}$, and the set inclusion relation $\subseteq$. There is a definition of commutativity of two elements for any orthomodular lattice that coincides with the usual commutativity of operators when applied to projectors in the Hilbert space. Two elements A and B commute when

$$A = (A \wedge B) \vee (A \wedge B^{\perp}) \,. \tag{16}$$

This definition coincides with the one given above for the commutativity of causally closed sets. Thus, the compatibility requirement that two causally closed sets have Cauchy surfaces which can be extended to a common global one can be put in the same mathematical terms as the compatibility condition for two projection operators to be simultaneously measurable. With this notation, the SSA and SSB relations for two commuting sets A and B are

[1] This equivalence holds for bounded sets. The causally closed sets $O = O^{\perp\perp}$ are exactly the domain of dependence of achronal sets bounded in time [22].

$$S(A)+S(B)\geq S(A\vee B)+S(A\wedge B)\,, \tag{17}$$
$$S(A)+S(B)\geq S(A\wedge B^{\perp})+S(B\wedge A^{\perp})\,, \tag{18}$$

respectively.

A relativistic entropy function is defined as a positive[2] function on the causally closed sets in Minkowski space, symmetric under Poincare transformations, and satisfying SSA and SSB for commuting pairs of subsets. The use of the relation SSB in the proofs below may be avoided, but it considerably simplify the demonstrations.

Thus, the problem of finding the functional dependence of the geometric entropy takes a precise mathematical form, being equivalent to find the most general relativistic entropy function. We note that a linear combination of solutions with positive coefficients is also a solution.

A similar problem can be stated in any space-time, but in absence of the Poincare symmetry the number of solutions can be very big. For example, the flux of any future directed conserved current through a Cauchy surface for a causally closed set A is always a solution of SSA and SSB since it satisfies the strong subadditive equation (rather than inequality). Of course this solution is not Poincare invariant in Minkowski space. Symmetries with orbits that are exclusively time-like (or space-like) do not impede this type of extensive solutions.

In this paper we are only using causally closed sets with compact closure, and, as in the Euclidean case, make no distinction between sets differing by zero measure subsets. The game to compare the entropies on different sets using strong subadditivity is more subtle than in an ordinary additive measure theory. Because of that we further restrict the class of subsets. To see what are the difficulties about, consider first the polyhedra on the Euclidean space $\mathbf{R}^d$. The intersections and unions of polyhedra are again polyhedra. Thus, the relations SSA and SSB can not be used to obtain information for the entropy of sets with curved borders from the entropy on polyhedra. In order to focus on the most restrictive relations coming from the entropy properties and the symmetries, in the following we consider the domain of the Euclidean entropy functions to be the class of all polyhedra. Note, however, that these can have an arbitrarily large number of faces as small as we want, and, in this sense, we can approach any set by polyhedra. To connect the results about polyhedra with other kind of sets some continuity condition is needed, and we do not deal with this problem here. In the relativistic $d+1$ dimensional case we consider the class of relativistic polyhedra, that is, spatial d dimensional sets (or their corresponding $d+1$ dimensional causally closed sets) formed by union of a finite number of polyhedra on spatial hyperplanes. These also form a closed class under union and intersection when taken between commuting sets.

A note on the terminology. We call $\mathrm{vol}(X)$ the volume of a d dimensional Euclidean polyhedron X, and $\mathrm{area}(X)$ to the area of the $d-1$ dimensional boundary of X. For a relativistic $d+1$ dimensional polyhedron X we call $\mathrm{area}(X)$ the area of its spatial corner, which is the boundary of any Cauchy surface for X. A relativistic entropy function S

[2] There was an indication of a negative geometric entropy when gauge fields are present [6]. This troubling result was finally corrected [8,10].

on $d+1$ dimensions when restricted to polyhedra on a unique hyperplane gives place to an Euclidean entropy function on d dimensions. This latter is independent of the chosen hyperplane. When it is possible, we identify these functions, and call them with the same letter.

IV. ENTROPY IN AN EUCLIDEAN SYMMETRIC SYSTEM

A. The one dimensional case

Consider a system on the real line and a translational invariant density matrix ρ. The connected compact sets on $\mathbf{R}^1$ are intervals, determined up to a translation by their length x. On these intervals the entropy can be written as a function on a real variable $S(x)$. The results we obtain here about $S(x)$ can be found for example in the ref. [15].

Take two intervals of length $(x+y)/2$, with x and y positive and $y>x$, and arrange them so that their intersection has length x and consequently their union has length y. Applying strong subadditivity we have

$$S\left(\frac{x+y}{2}\right) \geq \frac{1}{2}S(x)+\frac{1}{2}S(y)\,. \tag{19}$$

Suppose now that $S(x)$ is infinite. Then it follows from (19) that $S(z)$ must be infinite for $(x/2)\leq z\leq\infty$. Repeatedly using this argument we have that the function $S(x)$ is either finite for all x or always infinite.

Similar arguments can be used to show that all the entropy functions considered in the paper are either infinite or always finite. We are not coming back to this point and assume a finite S in the following.

The relation (19) for the function $S(x)$ is called weak concavity, while the concavity is the property

$$S(\lambda x+(1-\lambda)y)\geq \lambda S(x)+(1-\lambda)S(y)\,, \tag{20}$$

for any $0\leq\lambda\leq 1$. A function is concave when the segment joining any two points in its graph lies below the function curve (see the fig.(2)). Weak concavity is in fact very near concavity, since applying (19) repeatedly it is easy to obtain eq.(20) for any rational λ with a power of 2 as a denominator. For example, we have $S(3/4x+1/4y)\geq\frac{1}{2}S(x)+\frac{1}{2}S(1/2x+1/2y)\geq\frac{3}{4}S(x)+\frac{1}{4}S(y)$. To show that $S(x)$ is actually concave suppose that $x<z<y$ and, violating concavity,

$$S(z)=\frac{y-z}{y-x}S(x)+\frac{z-x}{y-x}S(y)-\delta\,, \tag{21}$$

with $\delta>0$. Let $w=\lambda x+(1-\lambda)y$ with $\lambda\in[0,1]$ a rational with a power of 2 as a denominator. We can take $w=z+\epsilon$ with ϵ a positive number as small as we want. Positioning two intervals of length z such that their union has length $z+\epsilon=w$, and their intersection has length $z-\epsilon$, we have from SSA and (21)

$$S(z-\epsilon)\leq S(z)-\delta+\mathcal{O}(\epsilon)\,. \tag{22}$$

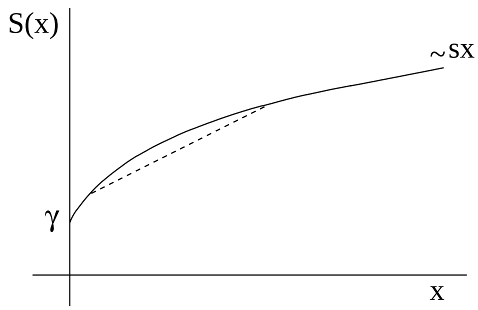

FIG. 2. A non decreasing concave function $S(x)$. Two points on the curve $S(x)$ determine a segment that lies below the function graph. The limit of $S(x)$ for $x \to 0$ is γ, and the curve slope approaches to s as x goes to infinity (however, $S(x)$ does not necessarily approach any straight line asymptotically).

Thus, as near to z as we want there are points in which, let say, $S < S(z) - \frac{1}{2}\delta$. By repeating this argument for a sufficient number of times we could arrive at the result that there would be points as near to z as we want that have negative S. As the entropy is positive, equation (21) must be wrong, and the function S is concave, equation (20) holds for any $\lambda \in [0,1]$. The converse is also true, concavity implies strong subadditivity for intervals.

Suppose that for some $y > x$ it is $S(y) < S(x)$. Then the function S is decreasing for any $z > y$, because a consequence of concavity is that the slope of the function S always decreases. This would lead to $S(z) < 0$ for high enough z. To avoid it the function S must be non decreasing everywhere. The monotonicity of the entropy appears here as a consequence of translation symmetry. However, order by inclusion is not warranted for the entropies of sets formed by several disjoint intervals, or connected sets in more dimensions. The concavity also imply that $S(x)$ is continuous.

Being nondecreasing and positive $S(x)$ has a well defined limit as x goes to zero. We call γ this limit. Given $x < y$ we have from concavity that

$$S(x) \geq \gamma + x\frac{S(y) - \gamma}{y} \,. \tag{23}$$

Since $\gamma \geq 0$ it follows that $S(x)/x$ is a non increasing function. This leads to the existence of the limit $\lim_{x\to\infty} S(x)/x = s$, with $s \geq 0$. Therefore there exist a well defined notion of mean entropy in the system (otherwise $S(x)/x$ could increase to infinity or oscillate indefinitely). For big enough sets the entropy is approximately extensive, at least if $s \neq 0$.

B. The limit for the entropy of small one dimensional sets

The measure zero sets have zero entropy. However, sequences of sets with measures converging to zero can have non zero entropy limit. The limit of $S(x)$ when x goes to zero is in general a positive number γ. Consider now the case of a set formed by two intervals of lengths x_1 and x_2 separated by a distance y_1. We are interested in the limit $l = x_1 + y_1 + x_2 \to 0$. If ϵ is a small positive number we can choose l such that $S(l) \leq \gamma + \epsilon$. Placing an interval of length $y_1' \leq l$, $y_1' \geq y_1$ as shown in fig.(3)a it follows from SSA that

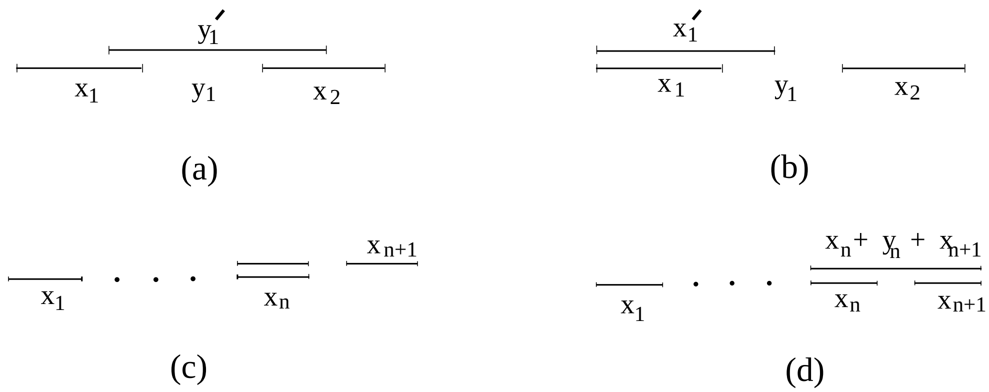

FIG. 3. Geometric constructions used to prove the formula (28) for the entropies of one dimensional sets in the limit of small size.

$$S(x_1, y_1, x_2) + S(y_1') \geq S(x_1', y_1, x_2') + S(l)\,, \tag{24}$$

with $x_1' \leq x_1$, $x_2' \leq x_2$, $x_1' + x_2' + y_1 = y_1'$. Taking into account the monotonicity of $S(x)$ we have

$$S(x_1, y_1, x_2) \geq S(x_1', y_1, x_2')\,, \tag{25}$$

for any $x_1' \leq x_1$ and $x_2' \leq x_2$. Similarly, from the same construction of fig.(3)a but now using SSB we have

$$S(x_1, y_1, x_2) \geq S(x_1', y_1', x_2') - \epsilon\,, \tag{26}$$

for any $y_1' \geq y_1$, $x_1' \leq x_1$, $x_2' \leq x_2$, and $x_1 + y_1 + x_2 = x_1' + y_1' + x_2'$. Thus, reducing the length of the two intervals reduces the entropy (neglecting contributions of order ϵ). The converse is given by the construction of figure (3)b. Applying strong subadditivity we get

$$S(x_1, y_1, x_2) \geq S(x_1', y_1', x_2) - \epsilon\,, \tag{27}$$

for any $x_1' \geq x_1$, $y_1' \leq y_1$, and $x_1 + y_1 = x_1' + y_1'$.

From (25), (26) and (27) we see that $S(x_1, y_1, x_2)$ converge to a limit δ when $x_1 + y_1 + x_2$ go to zero, and this limit is independent of the way in which the interval lengths x_1 and x_2 go to zero.

Now consider the entropy $S(x_1, y_1, ..., y_{m-1}, x_m)$ of a set formed by m disjoint intervals of lengths $x_1, ..., x_m$ separated by distances $y_1, ..., y_{m-1}$. We analyze the entropy in the limit when $l = x_1 + y_1 + ... + x_m$ go to zero and prove by induction that this depends only on the number of connected components m, and is given by

$$\lim_{l \to 0} S(x_1, y_1, ..., y_{m-1}, x_m) \equiv S_m = \alpha + \beta m\,, \tag{28}$$

where

$$\alpha = 2\gamma - \delta\,, \tag{29}$$
$$\beta = \delta - \gamma\,. \tag{30}$$

For $m = 1$ and $m = 2$ we have proved this formula. Suppose it is valid for $m \leq n$. From the constructions of fig.(3)c and fig.(3)d and SSA we have

$$S(x_1, y_1, ..., y_{n-1}, x_n) + S(x_n, y_n, x_{n+1}) \geq S(x_1, y_1, ..., y_n, x_{n+1}) + S(x_n)\,, \tag{31}$$

$$S(x_1, y_1, ..., y_n, x_{n+1}) + S(x_n + y_n + x_{n+1}) \geq \tag{32}$$

$$S(x_n, y_n, x_{n+1}) + S(x_1, y_1, ..., y_{n-1}, x_n + y_n + x_{n+1})\,, \tag{33}$$

respectively. Taking the limit of small $x_1 + y_1 + ... + y_n + x_{n+1}$ in these inequalities and using the induction hypothesis it follows that (28) is also valid for $m = n + 1$. Then it holds for any integer $m \geq 1$.

It is immediate from subadditivity and from (28) that

$$0 \leq \gamma \leq \delta \leq 2\gamma\,. \tag{34}$$

These relations for the parameters simply translates into the positivity of the independent variables α and β, which do not have to satisfy any constrain. We will also write (28) as

$$S_m = \gamma + (m-1)\beta\,, \tag{35}$$

where $\gamma \geq \beta \geq 0$.

C. The multidimensional case

Consider now a translation invariant state in $\mathbf{R}^d$. Let the unit vectors $v_1, ..., v_d$ form an orthogonal basis of $\mathbf{R}^d$. Define the family of rectangular polyhedra as all the translates of the $R(a_1, ..., a_d) = \{x/x = \sum_{i=1}^{d} \lambda_i v_i, \lambda_i \in [0, a_i]\}$, where $(a_1, ..., a_d)$ are the sides lengths. We often use just a symbol R for these polyhedra, what should not be confused with the symbol for the reals $\mathbf{R}$. The entropy can be written as a function on d variables $S(a_1, ..., a_d)$ on these sets. The same constructions made above for the one dimensional case apply here on each direction separately. Therefore $S(a_1, ..., a_d)$ is a non decreasing and concave function in each a_k. Then the entropy is ordered by inclusion among the rectangular polyhedra. Also, by concavity $S((a_1, ..., a_d)/(a_1 \times ... \times a_d)$ is non increasing in each a_k separately. Then the entropy divided the d dimensional volume for has a limit for the rectangular polyhedra when the volume goes to infinity along any given increasing sequence R_n. When all the sides of R_n go to infinity, the limit s of entropy over volume must be the same number independently of the particular sequence. This is because giving two of such sequences of rectangular polyhedra with sides going to infinity, every element of one sequence must be smaller than some element of the other sequence, and in consequence none of the limits of the entropy over volume could be bigger than the other. Thus s is really the entropy per unit d dimensional volume of the system. If the volume goes to infinity but one of the sides remains bounded the limit of entropy per unit volume could differ from s. Physically, this means that the effects of the boundary may always be relevant in this case.

When in addition to symmetry under translations there is rotational symmetry this result can be generalized to different sequences of polyhedra. Any sequence where the volume goes to infinity at a greater rate than the bounding area (going to infinity in the sense of Van Hove), and where the number of faces remains bounded, has the limit s of entropy over

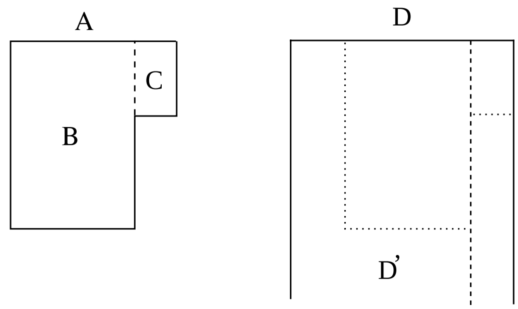

FIG. 4. Eating from inside a strip D to a smaller strip D' (the right side of D' is the dashed line), by means of $A-B$. One has to position A inside D (shown with the dotted line) and take out from D the sector corresponding to $A-B=C$. Then translating down A inside D and repeating the operation it is possible to eat all $D-D'$.

volume, which is independent of the particular sequence [15]. It is in this type of sequences that one expects the surface terms contribution to the entropy to be negligible in the limit. We do not use this result here. Instead we are more interested in the properties of the entropy function for sets with finite area and vanishing volume than in the infinite volume limit. These flat sets may not be so interesting in statistical mechanics but they are crucial for the relativistic geometric entropy. The boost symmetry allows to amplify their width and reduce most of the general relativistic problem to the case of flat Euclidean sets.

D. Entropy for flat sets

Let us consider polyhedra on $\mathbf{R}^d$ of the form $X \times [x_1, y_1, ..., y_{m-1}, x_m]$, where X is a $d-1$ dimensional polyedron and $[x_1, y_1, ..., y_{m-1}, x_m]$ is a set with m connected components in $\mathbf{R}^1$. Here $x_1, ..., x_m$ are the single component lengths and $y_1, ..., y_{m-1}$ are the separation distances between adjacent components. Call the corresponding entropies $S(X, x_1, y_1, ..., x_m)$. For a fixed X these polyhedra form a translation invariant one dimensional system, and we have from equation (35) that

$$\lim_{x_1+y_1+...+x_m \to 0} S(X, x_1, y_1, ..., x_m) = \gamma(X) + (m-1)\,\beta(X)\,, \tag{36}$$

where $\gamma(X)$ and $\beta(X)$ are positive and Euclidean invariant functions on polyhedra in $\mathbf{R}^{d-1}$, and $\gamma(X) \ge \beta(X)$. It follows from SSA and SSB in d dimensions that $\gamma(X)$ and $\beta(X)$ also satisfy SSA and SSB. Note that we use the same symbols β and γ that represent constants in the one dimensional case for denoting functions in the multidimensional case.

In addition, these Euclidean entropy functions are also ordered by inclusion. To show it we first introduce a method that we can call the 'eating from inside' procedure. This was developed in [13] as a tool to prove certain monotonicity relations.

Suppose we have two sets A and B with $B \subset A$, and we want to obtain information about the difference of their entropies. Call $C = A-B$. At our convenience we choose a set D with $A \subset D$. Then from strong subadditivity it follows that $S(A) + S(D-C) \ge S(D) + S(B)$. This can be written as $S(A) - S(B) \ge S(D) - S(D-C)$. Now, if a symmetric copy A_1 of A can be situated inside $D-C$ we can repeat the procedure with $D-C$ in the place of C,

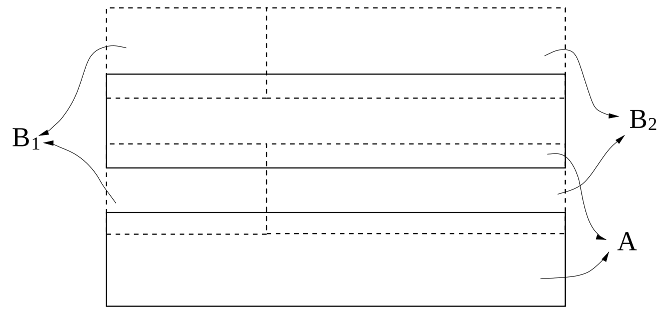

FIG. 5. The geometric construction used to show that the function β is proportional to the volume. The solid line represents a transversal view of the two component set A. Drawn with dashed lines are the two component set B_1 on the left and the two component set B_2 on the right. All the connected components of these sets have the same height a.

leading to $S(A)-S(B) \geq S(D-C)-S(D-C-C_1)$, where C_1 is the symmetric copy of C corresponding to the same transformation that carries A into A_1. Summing up this and the previous inequality we have that $2(S(A)-S(B)) \geq S(D)-S(D-C-C_1)$. The procedure can be continued if at the n^{th} step we can place a copy of A inside what is left from D after subtracting all the previous symmetric the copies of C (see fig.(4)). In that case we have

$$S(A)-S(B) \geq \frac{1}{n}\left(S(D)-S(D')\right) , \tag{37}$$

where $D'=D-C-C_1-...-C_{n-1}$. In some situations the right-hand side in (37) is known, specially if the limit $n \to \infty$ can be taken. From this we can obtain information about the difference of entropies of two sets A and B ordered by inclusion. We say that to obtain (37) we eat from inside D to D' by means of $A-B$.

Take now any $d-1$ dimensional polyhedra X and Y with $Y \subseteq X$, and let the one dimensional set $Z=[x_1,y_1,...,y_{m-1},x_m]=[a,a,...,a,a]$ be formed by m intervals of length a, with separation distance a between adjacent intervals. Consider the sets $A=X\times Z$, $B=Y\times Z$, $C=(X-Y)\times Z$, and $D=X\times[2maN]$. Here $[l]$ means a single interval of length l, and N is a big integer. Then it is easy to see that translating A, B, and C on the direction perpendicular to X and Y we can eat from inside $2N$ copies of C from D, leading to the remaining set $D'=Y\times[2maN]$. Application of (37) gives

$$S(A)-S(B) \geq \frac{ma}{(2maN)}\left(S(D)-S(D')\right) . \tag{38}$$

The results for one dimensional translation invariant systems imply that $S(U\times[l])/l$ has a limit s_U for l going to infinity. This implies taking $N\to\infty$ that

$$S(A)-S(B) \geq ma\,(s_X - s_Y) . \tag{39}$$

Now, taking the limit $a \to 0$ we have from this inequality that $\gamma(X)+(m-1)\beta(X)$ is ordered by inclusion for any $m\geq 1$. Consequently $\gamma(X)$ and $\beta(X)$ are ordered by inclusion.

Moreover, the functional form of $\beta(X)$ can be calculated explicitly. Let us take $A=X\times Z$ with $Z=[x_1,y_1,x_2]=[a,a/2,a]$ and X a $d-1$ dimensional polyhedron. Also take $B_1=Y_1\times Z$ and $B_2=Y_2\times Z$ with Y_1 and Y_2 two $d-1$ dimensional polyhedra such that

$X = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$. Suitably placing B_1 and B_2 to have empty (or measure zero) intersection and to cover the gaps on A (see fig.(5)), we can arrange that the union $A \cup B_1 \cup B_2 = X \times [(13/4)a]$ and the intersections $A \cap B_1 = Y_1 \times Z'$, $A \cap B_2 = Y_2 \times Z'$, where $Z' = [x_1, y_1, x_2, y_2, x_3] = [a/4, a/2, a/4, a/2, a/4]$. Then, from strong subadditivity for the pairs (A, B_1) and $(A \cup B_1, B_2)$, and taking the limit $a \to 0$ we get

$$\beta(X) \geq \beta(Y_1) + \beta(Y_2)\,. \tag{40}$$

This inequality is opposite to the subadditive inequality for β since $X = Y_1 \cup Y_2$. Thus the equation holds,

$$\beta(X) = \beta(Y_1) + \beta(Y_2)\,. \tag{41}$$

With N^{d-1} copies of a rectangular polyhedra R with sides $(l_1, ..., l_{d-1})$ we can exactly cover a rectangular polyhedron U with sides $(N\,l_1, ..., N\,l_{d-1})$. Using (41) it is

$$\beta(R) = \mathrm{vol}(R)\,\frac{S(U)}{\mathrm{vol}(U)}\,. \tag{42}$$

The fraction on the right hand side converges to a limit $2s$ for large N. The factor 2 is chosen for future convenience. Thus, we have that the function β is proportional to the volume for any rectangular polyhedron. To show that $\beta(X)$ is proportional to the volume for any polyhedron X divide the space $\mathbf{R}^{d-1}$ with a cubic grid formed by copies of the cubic element A. Call $\Gamma^-(A, X)$ the union of elements of the grid that are included in X, and $\Gamma^+(A, X)$ the union of elements of the grid having non empty intersection with X. Using (41) and that β is proportional to the volume on cubes it follows that the entropies $S(\Gamma^\pm(A, X)) = 2s\,\mathrm{vol}(\Gamma^\pm(A, X))$ are proportional to the volumes. Then, using the order by inclusion of β, the fact that $\mathrm{vol}(\Gamma^-(A, X)) \leq \mathrm{vol}(X) \leq \mathrm{vol}(\Gamma^+(A, X))$, and that these volumes coincide in the limit of small A, we obtain

$$\beta(X) = 2s\,\mathrm{vol}(X) \tag{43}$$

for every X. Note that the only symmetry we used was translational invariance.

The function $\alpha = \gamma - \beta$ is positive and strong subadditive because $\beta(X) = 2s\,\mathrm{vol}(X)$ satisfies the strong additive equality, $\beta(A) + \beta(B) = \beta(A \cap B) + \beta(A \cup B)$.

Summarizing, we have shown the following theorem on the entropy limit for flat sets

Theorem 1: For an Euclidean invariant d dimensional system the limit of the entropy for sets of the type $X \times [x_1, y_1, ..., x_m]$ as all the $x_1, y_1, ..., x_m$ go to zero exists and is given by

$$\lim_{x_1+y_1+...+x_m \to 0} S(X, x_1, y_1, ..., x_m) = \gamma(X) + (m-1)\,2s\,\mathrm{vol}(X)\ , \tag{44}$$

where $\gamma(X)$ is an ordered by inclusion $d-1$ dimensional Euclidean entropy function, and $\gamma(X) \geq 2s\,\mathrm{vol}(X)$.

One could wonder about the entropy limits for lower dimensional objects. Define $\hat{\gamma}$ and $\hat{\beta}$ as the operators that transform Euclidean d dimensional entropy functions into Euclidean

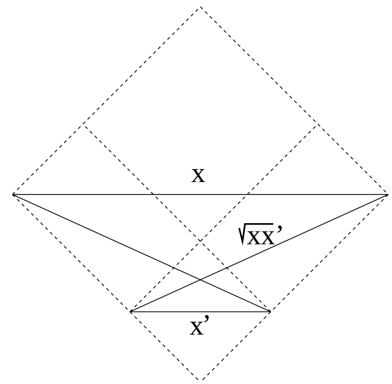

FIG. 6. Two commuting sets with straight Cauchy surface size $\sqrt{xx'}$ in 1+1 dimensions intersect in another diamond-like set of size x', and their union is, after causal completion, a diamond set of base size x.

$d-1$ dimensional entropy functions and whose action is defined as $\hat{\gamma}_S = \hat{\gamma}(S) = \gamma$, $\hat{\beta}_S = \hat{\beta}(S) = \beta$, with γ and β as above. The monotonicity of γ and β imply that $\hat{\beta} \circ \hat{\gamma} = 0$, and $\hat{\beta} \circ \hat{\beta} = 0$, and the explicit form of β leads to $\hat{\gamma} \circ \hat{\beta} = 0$. However, the composition $\hat{\gamma} \circ \hat{\gamma}$ can be different from zero when applied to a general entropy function S. We call $\hat{\gamma}^n_S$ for $n = 0, ..., d$ the composition of S with n times the operator $\hat{\gamma}$, and define $\hat{\gamma}^0_S = S$.

V. GEOMETRIC ENTROPY IN MINKOWSKI SPACE

A. The $1+1$ dimensional case

Before embarking in the calculation of the functional form of the entropy in the general $d+1$ dimensional Minkowski space we focus on the simpler $1+1$ case. This allows to see without the complications of higher dimensions the important role played by the boost symmetries. The result for the entropy and the method to obtain it are similar to the ones for the 1 dimensional Euclidean small sets.

The connected causally closed sets are determined in 1+1 dimensions by a single number x, the size of the unique straight Cauchy surface (the size of the diamond base, see fig.(6)). The corresponding entropy $S(x)$ is a positive, non decreasing and concave function because of translation symmetry. As above, call γ the constant $\lim_{x\to 0} S(x)$.

The additional boost symmetry greatly constrain the function $S(x)$. To see this, consider the construction of fig.(6). Given $x' \leq x$ we place two commuting boosted one component sets of size $\sqrt{x\,x'}$ in such a way their intersection is a one component set of size x' and their union is another one component set of size x. The application of SSA gives

$$S(x) + S(x') \leq 2\, S(\sqrt{x\,x'})\,. \tag{45}$$

Making x' as small as we want in this inequality we prove that $S(x) \leq \gamma$. As $S(x)$ is non decreasing this means that $S(x) \equiv \gamma$ is constant. Due to the boost symmetry we can form a Cauchy surface for a given diamond D using two boosted diamonds of size as small as we want disposed along the null surfaces of D. This, and the subadditive law, is what impedes an entropy increasing like the volume.

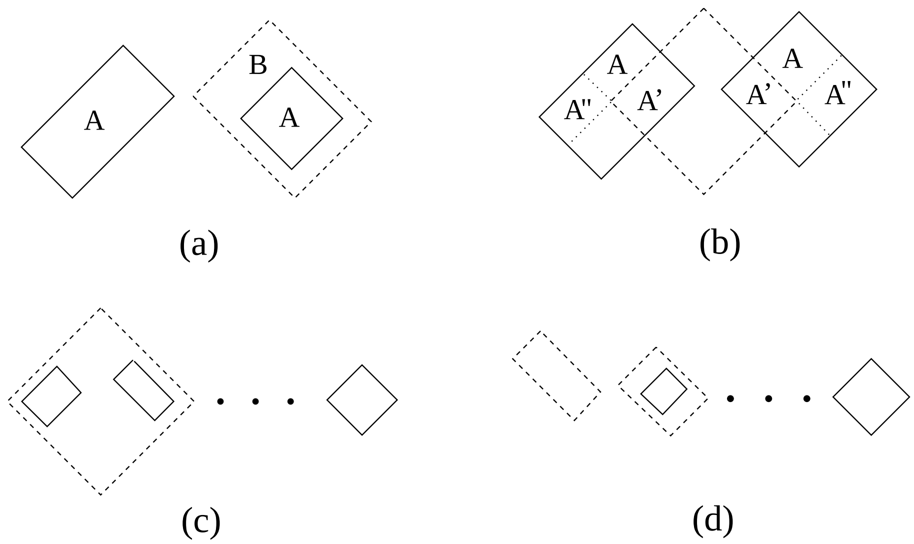

FIG. 7. Two dimensional constructions used to prove Theorem 2. In (a), (c) and (d) it is considered the SSA between the solid line and the dashed line set. In (b) both the SSA and the SSB between the solid line set and the central dashed diamond are used.

The entropy for sets consisting on several components spatially separated can in principle depend on the lengths of the individual diamond bases, their separation distances, and relative angles.

Let us start with the case of two component sets. In the construction of fig.(7)a the union of a two component set A with a single diamond B is a set A', having two components, one of which is bigger than the corresponding component in A. Application of SSA gives

$$S(A)+\gamma \geq S(A')+\gamma\,. \tag{46}$$

Thus, the entropy decreases with increasing size of the individual connected components. Now, the application of SSA and SSB to the construction of fig.(7)b leads to the inequalities

$$S(A)+\gamma \geq S(A')+\gamma\,, \tag{47}$$
$$S(A)+\gamma \geq S(A'')+\gamma\,, \tag{48}$$

where A' and A'' are both included in A but A' shares with A the internal spatial corners and A'' shares with A the external spatial corners. Combining these relations we can prove that the entropy of any two component set X is greater than the entropy of any other two component set X' where each individual component of X' is included in a corresponding component of X. Thus, the entropy of the two component sets is increasing and decreasing with inclusion, and then it must be a constant. Call δ this constant.

For a greater number of connected components the entropy can be expressed in terms of γ and δ and we show the following

Theorem 2: The most general form for a relativistic entropy function in $1+1$ dimensional Minkowski space is given by

$$S(Z)\equiv S_m=\gamma+(m-1)\,\beta\,, \tag{49}$$

where m is the number of connected components of Z and $\gamma \geq \beta \geq 0$.

In terms of δ it is $\beta = \delta - \gamma$. The proof is very similar to the corresponding one for the small one dimensional sets in Section III. To show (49) by induction on the number of components note that for $m = 1$ and $m = 2$ this equation holds. Supposing it is valid for $m \leq n$ we have from the constructions of fig.(7)c and fig.(7)d

$$S_{n+1}(X) + \gamma \geq S_n + \delta \,, \tag{50}$$

$$S_n + \delta \geq S_{n+1}(X) + \gamma \,. \tag{51}$$

This proves the formula (49). It follows from positivity and subadditivity that the parameters have to satisfy the relations

$$0 \leq \gamma \leq \delta \leq 2\,\gamma \,, \tag{52}$$

$$0 \leq \beta \leq \gamma \,. \tag{53}$$

The meaning of these constraints is clarified if we write S_m as a sum

$$S_m = \alpha + \beta\, m \,, \tag{54}$$

of two independent solutions with independent and arbitrary positive coefficients. Here $\alpha = \gamma - \beta$. One of the solutions is the constant term and the other is topological, being proportional to the number of connected components m. It is easy to check that (54) does indeed satisfy SSA and SSB for two arbitrary commuting sets. Thus, this formula, or eq. (49), represents the most general solution for the entropy in $1+1$ dimensional Minkowski space.

B. Multidimensional case $d > 1$

In this Section we show the theorem

Theorem 3: The most general form for a relativistic entropy function on the relativistic polyhedra on $d+1$ dimensional Minkowski space, with $d > 1$, is given by

$$S(X) = \alpha_0 + s \;\text{area}(X) \,, \tag{55}$$

where α_0 and s are non negative constants and area(X) is the area of the spatial corner of X.

We divide the proof in several parts.

1. Relativistic Lemmas

To apply SSA or SSB between two sets we have to ensure that they commute, what complicates the proofs for $d > 1$. In the following we incorporate into the Euclidean function given by restricting the relativistic entropy function to a single spatial hyperplane some

information coming from the boost symmetry. With these additional properties at hand we will be able to work in Euclidean space where all sets commute.

An example where commutativity fails is given by adding one perpendicular spatial dimension to the sets in figure (6). The two boosted intervals of length $\sqrt{x\,x'}$ there can be seen as the transversal section of two boosted rectangles in $2+1$ dimensional Minkowski space. However, these rectangles do not commute when they are in such a position since their boundaries are not spatial to each other.

There is a limit where commutativity can be applied in an effective form. Consider the product $R \times Z$ of a big $d-1$ dimensional rectangular polyhedron R and a relativistic polyhedron Z in $1+1$. We mean here by the product of a polyhedron R with a relativistic polyhedron Z the relativistic polyhedron generated by the Cartesian product of R with a Cauchy surface for Z, since the Cartesian product of R with the relativistic polyhedron corresponding to Z is not a relativistic polyhedron on $d+1$. Given two commuting $1+1$ dimensional sets Z and Z' the sets $R\times Z$ and $R\times Z'$ may not commute. However, taking two fixed commuting Z and Z', we can construct a relativistic polyhedron U commuting with $R\times Z$ by deforming $R\times Z'$ near the borders of R. More precisely, we take U as the domain of dependence of the union of $(R\times Z')\cap(R\times Z)$ and $(R\times Z')\cap(R\times Z)^{\perp}$. Here $A^{\perp}$ is the set formed by all the points spatially separated from A. When the sides of R are big, U differs from $R \times Z'$ by a set located along the boundary of R, and whose transversal size remains fixed if the sizes of R are increased. Thus, the difference in entropy between $R \times Z'$ and U can be bounded by a number proportional to the border area of R by using intersections and unions of rectangular polyhedra with bounded sides. If we consider the limit of the entropy $S(R\times Z)$ divided by the volume $\mathrm{vol}(R)$ for the sets of the type $R\times Z$ with R going to infinity the terms growing like the border area of R can be neglected. Therefore, in this limit we can use commutativity between the sets $R \times Z$ and $R \times Z'$ with commuting Z and Z', and use the two dimensional results. We have the

Lemma 1: Let R be a $d-1$ dimensional rectangular polyhedron and Z a $1+1$ relativistic polyhedron then

$$\lim_{R\to\infty} \frac{S(R \times Z)}{\mathrm{vol}(R)} = \gamma_\infty + 2s\,(m-1)\,, \tag{56}$$

where m is the number of connected components in Z, s is defined by Theorem 1, and γ_∞, with $\gamma_\infty \geq 2s$, is the limit $\gamma(R)/\mathrm{vol}(R)$ for R going to infinity.

The explicit form of the right hand side follows from Theorem 1, since the two dimensional solution tell us that this term must be a constant plus a constant times the number of components of Z. Then we can calibrate the parameters from the special case of Z lying in a single spatial line, $Z = [x, y, ..., y, x]$, for small x, y.

In particular the Lemma applies to the sets $Z = [x_1, y_1, ..., x_m]$ that lie in a single spatial plane, with arbitrary $x_1, y_1, ..., x_m$. Thus, it removes the constrain of small $x_1 + y_1 + ... + x_m$ in Theorem 1, in the limit of R going to infinity. Of course, this includes information about boost symmetry and it is not the case for a general Euclidean function.

With the notation of Section IV consider now the entropies $S(R, x_1, y, x_2)$ for the sets $X = R \times [x_1, y, x_2]$ contained in a single spatial hyperplane, and where R is a fixed $d-1$ dimensional rectangular polyhedron. Call $X_1 = R \times [x_1]$ and $X_2 = R \times [x_2]$. Theorem

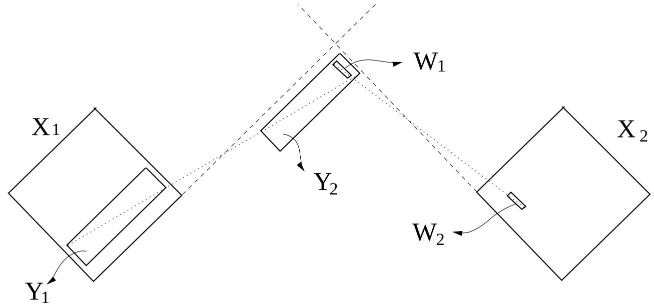

FIG. 8. Two dimensional cut of the geometrical construction used to prove inequality (58). The lines at 45° are null surfaces. The dotted lines represent the spatial hyperplanes containing the defining Cauchy surfaces for $Y = Y_1 \cup Y_2$ and $W = W_1 \cup W_2$.

1 showed the form of this function for the limit in which all x_1, y, x_2 go to zero. We can do better in the relativistic case, and find the limit for x_1 and x_2 going to zero and any y. The first inequality we obtain comes eating from inside $U \times [x_1, y, x_2]$, where U is a big rectangular polyhedron, to $U \times [x_1]$ by means of $X - X_1$. Using the Lemma 1 and Theorem 1 this leads to

$$\lim_{x_1+x_2\to 0} S(R \times [x_1, y, x_2]) \geq \gamma(R) + 2s\,\mathrm{vol}(R)\,. \tag{57}$$

Then consider the following construction shown in the figure (8) through a two dimensional projection. The set $Y = Y_1 \cup Y_2$ is of the form $R' \times [x_1', y', x_2']$, but it appears highly boosted with respect to X in the direction perpendicular to R. The same happens for $W = W_1 \cup W_2 = R'' \times [x_1'', y'', x_2'']$. The rectangular polyhedron $R'' \subseteq R' \subseteq R$, the transversal sizes $(x_1', y', x_2', x_1'', y'', x_2'')$, and the boost factors are chosen in such a way that $Y_1 \subseteq X_1$, $W_1 \subseteq Y_2$, $W_2 \subseteq X_2$, and Y_2 is spatial to X_1 and X_2. We are taking then the limit of small $(x_1, x_2, x_1', y', x_2', x_1'', y'', x_2'')$ but keeping y fixed. In this limit the sides of R' and R'' can be taken to converge to those of R. Applying successively SSA to the pairs of sets (X_1, Y), $(X_1 \cup Y_2, W)$, and $(X_1 \cup Y_2 \cup W_2, X_2)$ we have

$$S(X \cup Y_2) \leq 2\gamma(R) - \gamma(R'') + 2s\,\left(\mathrm{vol}(R') + \mathrm{vol}(R'')\right) + \epsilon\,. \tag{58}$$

for ϵ as small as we want. Now, we eat from inside a three component set formed by the product of a big rectangular $d-1$ dimensional set U and the relativistic $1+1$ set given by the transversal projection of $X \cup Y_2$ (the set shown in figure (8)) to $U \times [x_1, y, x_2]$ by means of $(X \cup Y_2) - X$. Taking into account Lemma 1 this leads to

$$S(X \cup Y_2) - S(X) \geq 2s\,\mathrm{vol}(R')\,. \tag{59}$$

Combining the last two equations, taking a limit in which R'', R', and R all have the same volume and shape, and using the continuity of γ as a function of the sides of the rectangular polyhedron, we obtain the opposite inequality to (57). Then, we have

$$\lim_{x_1+x_2\to 0} S(R \times [x_1, y, x_2]) = \gamma(R) + 2s\,\mathrm{vol}(R)\,. \tag{60}$$

This equation allows us to prove the following

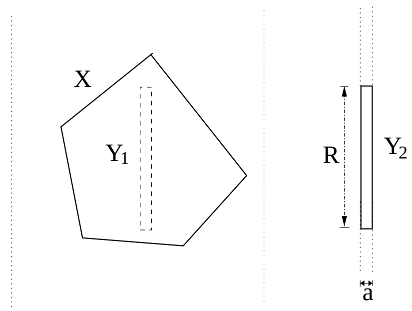

FIG. 9. The entropy of $X \cup Y_2$ is equal to $S(X) + 2s\,\mathrm{vol}(R)$ in the limit of small a. The dotted lines represent a two component strip-like set that can be eaten from inside down to a one component strip by means of $((X \cup Y_2) - X)$.

Lemma 2: Let X be a d dimensional polyhedron, R a $d-1$ dimensional rectangular polyhedron, $Z = [a, y, a]$, $Y = R \times Z$, and Y_1 and Y_2 the connected components of Y, symmetric copies of $R \times [a]$. Suppose that $Y \cap X = Y_1$ and that there is an open set O including Y_2 such that any translation of O in a plane parallel to R has no intersection with X (see fig.(9)). Then

$$\lim_{a \to 0} S(X \cup Y_2) = S(X) + 2s\,\mathrm{vol}(R)\,. \tag{61}$$

The proof uses the previous result (60) and Lemma 1. Applying SSA to the pair (X, Y) we have

$$\lim_{a \to 0} S(X \cup Y_2) \leq S(X) + 2s\,\mathrm{vol}(R)\,. \tag{62}$$

The opposite inequality follows from eating from inside a two component large rectangular strip-like set, one component containing X, the other having transversal size equal to a and containing Y_2, as show in fig.(9).

2. Subtraction of the constant term

Here we show that the limit α_0 of the entropy for rectangular polyhedra with vanishing small sides can be identified with the constant solution component of the general solution. In other words we show the following

Proposition 1: Let S be a relativistic entropy function. Then $S = \alpha_0 + S^1$, where α_0 is the limit value of the entropy for rectangular polyhedra of vanishing small sides, and S^1 is a relativistic entropy function with zero entropy limit for the rectangular polyhedra with vanishing small sides.

We know that the general solution must have a constant term. However, the identification of it with α_0 and the subtraction of α_0 are not automatic, since we have to show that S^1 is still positive and satisfies SSA and SSB. An example where this is not the case is given by the two dimensional solution (49). If one subtracts the entropy limit for small intervals (γ in the two dimensional case), the result is a positive but not strongly subadditive function.

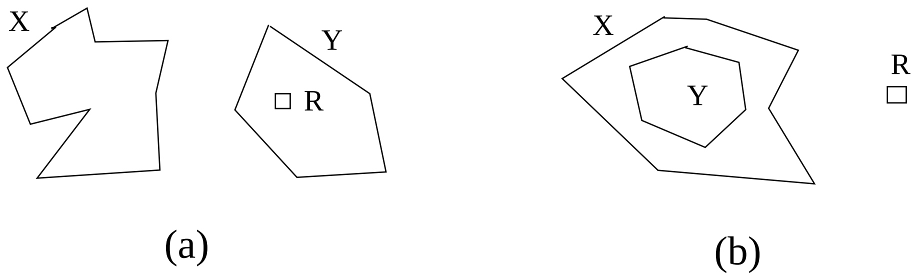

FIG. 10. (a)- The entropies of two non intersecting sets X and Y satisfy the relation (65). This is shown by constructing $X' = X \cup R$ and using SSA on the pair (X', Y). (b)- For $Y \subseteq X$ we can show that $S(X) + S(Y) \geq S(X - Y) + \alpha_0/2$ by using SSB on the pair (X', Y), where $X' = X \cup R$ and R is a small rectangular polyhedron spatially separated from X.

This is because subtracting γ is different from subtracting the constant solution. There is a non constant solution, the one proportional to β, that is constant over the one component sets and adds to γ. However, as it will be clear from what follows, in more dimensions there is no such solution. In particular, there are no positive topological solutions to the strong subadditive inequalities. Curiously, the Euler number $E(X)$, which is the second solution appearing in the two dimensional formula (54), satisfies the strong additive equality, $E(A) + E(B) = E(A \cup B) + E(A \cap B)$ in any dimension. However, for $d > 1$ it attains negative values.

Now consider any relativistic polyhedron X. It contains a sufficiently small rectangular polyhedron R' lying in a spatial hyperplane $\mathcal{P}$. We can generate a copy R of R' by translating R' on $\mathcal{P}$ in a direction perpendicular to a phase F of R', and such that $R \cap X = \emptyset$. For R sufficiently far from X there is a time-like hyperplane parallel to F in between X and R that divides the Minkowski space in two parts, one of which contains X and the other R. Let us call $X' = X \cup R$ (see fig.(10)a). Now, from Lemma 2 we have that for small enough R

$$S(X') - \epsilon \leq S(X) \leq S(X') + \epsilon \,. \tag{63}$$

Then, by means of a π angle rotation and a small translation on $\mathcal{P}$ we can make a copy of X' whose intersection with X' is a small rectangular polyhedron inside R. It follows from strong subadditivity that $S(X') \geq (1/2)\,\alpha_0$. Then taking the limit of small R we have

$$S(X) \geq \frac{1}{2}\alpha_0 \tag{64}$$

for any X.

Thus, we can safely subtract $(1/2)\alpha_0$ from the entropy without violating positivity. Strong subadditivity for the subtracted function will be automatic for the case where the two commuting sets X and Y have non empty intersection. When the intersection $X \cap Y = \emptyset$ we have to prove

$$S(X) + S(Y) \geq S(X \cup Y) + \frac{\alpha_0}{2} \,, \tag{65}$$

to show that the subtracted function is still subadditive. To see this, consider a set $X' = X \cup R$ as above, but now we choose R included in Y (see fig(10)a). Then

$$S(X') + S(Y) \geq S(X \cup Y) + S(R)\,. \tag{66}$$

From (63) and taking limits the relation (65) follows.

This shows that subtracting the constant solution $\alpha_0/2$ leads to a positive strongly subadditive solution. It also satisfies SSB as follows from similar arguments based on the construction of fig(10)b for the case in which the inequality is applied to the sets X and Y with $Y \subseteq X$ (otherwise there are two non empty sets on each side of the SSB inequality, which holds trivially for the subtracted function).

Once we have subtracted $\alpha_0/2$ we can repeat the procedure with the subtracted function $S - \alpha_0/2$ to subtract $\alpha_0/4$, and then $\alpha_0/8$, etc.. It follows that we can subtract a constant solution $x\alpha_0$ for any $0 < x < 1$ and still get a solution. Thus $S - \alpha_0$ is also positive and satisfies SSA and SSB.

3. $\gamma(X)$ is proportional to the volume

Taking into account the Proposition 1 we work with the subtracted function and assume $\alpha_0 = 0$. We show that $\gamma(X) = \beta(X) = 2s\,\mathrm{vol}(X)$, or, what is the same, $\alpha(X) = 0$. To do so we need to make some geometrical constructions and use that the entropy for sets that are flat in more than one perpendicular direction can be safely neglected. In the notation of Section III this writes $\hat{\gamma}^n(S) = 0$ for $n = 2, ..., d$. It is possible to write this last equation and the result we are looking for, $\gamma(X) = \beta(X)$ in a compact manner, that is, $\hat{\alpha}_S^n = (\hat{\gamma} - \hat{\beta}) \circ \hat{\gamma}_S^{n-1} = 0$, for $n = 1, ..., d$. Here we used that $\hat{\beta} \circ \hat{\gamma}_S^{n-1} = 0$ for $n \geq 2$. Recall that $\hat{\gamma}_S^0 = S$. Each $\hat{\alpha}_S^n$ is a function on the polyhedra on $\mathbf{R}^{d-n}$, with $\hat{\alpha}_S^1 = \alpha$ and $\hat{\alpha}_S^d = \alpha_0 = 0$.

Then, we show by induction that all the functions $\hat{\alpha}_S^n$ for $n = 1, ..., d$ are zero in the relativistic case when $\alpha_0 = 0$. For $n = d$ this is true. We show that if $\hat{\alpha}_S^{n+1} = 0$ then $\hat{\alpha}_S^n = 0$.

Consider first a polyhedra T in $\mathbf{R}^{d-n+1}$ with a wedge-like form, $T = \{(x_1, ..., x_{d-n+1})/0 \leq x_1 \leq \mu x_2 + \nu, 0 \leq x_2 \leq a_2, ..., 0 \leq x_{d-n+1} \leq a_{d-n+1}\}$. Call R the $d-n$ dimensional rectangular polyhedron with sides $(a_2, ..., a_{d-n+1})$. We are interested in the limit of $\hat{\gamma}_S^{n-1}(T)$ as μ and ν go to zero. Apply SSA and SSB to the constructions of fig(11)a and fig(11)b respectively, and using the induction hypothesis neglect sets that are small in more than one direction. We obtain two inequalities that are opposite in the limit of zero μ, ν. They lead to the equation

$$\lim_{\mu,\nu \to 0} \hat{\gamma}_S^{n-1}(T) = \hat{\gamma}_S^n(R)\,. \tag{67}$$

Thus this limit is the same as the one for flat rectangular polyhedra. In particular this holds for the function S of flat wedge sets, what corresponds to $n = 1$ in the last equation.

Now consider the construction of fig.(12) involving $d - n + 1$ dimensional polyhedra. The set X_1 is formed by the union of two copies of a flat rectangular polyhedron $U = R \times [a]$ having principal planes that form a small angle θ. Using the Lemma 2 we have that $\hat{\gamma}_S^{n-1}(X_2) \to \hat{\gamma}_S^{n-1}(X_1) + 2s\,\mathrm{vol}(R)\,\delta_1^n$ in the limit of small a (see fig. (12)b). By the same reason $\hat{\gamma}_S^{n-1}(X_2) \to \hat{\gamma}_S^{n-1}(X_3) + 2s\,\mathrm{vol}(R)\,\delta_1^n$ in that limit (see fig.(12)c). Then it follows that $\lim_{a\to 0} \hat{\gamma}_S^{n-1}(X_1) = \lim_{a\to 0} \hat{\gamma}_S^{n-1}(X_3)$. This means that we can move each of the components that are copies of U in X_1 a direction perpendicular to their principal

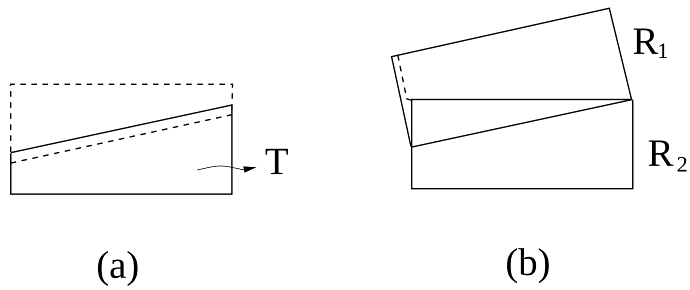

FIG. 11. Geometrical constructions used in the text to show that the limit of the entropy for a flat wedge-like $d-n+1$ dimensional set T coincides with $\hat{\gamma}_S^n(R)$, where R is the rectangular polyhedron forming the big phase of T. The picture shows a transversal view, where R is reduced to an interval. (a)- Two copies of T are placed such that their intersection and union are rectangular flat polyhedra with big phase R (neglecting sets small in more than one perpendicular direction). (b)- Two rectangular flat polyhedra R_1 and R_2 are placed such that both $R_1 - R_2$ and $R_2 - R_1$ are equal to T modulo sets that are small in more than one direction.

plane without changing the limit of the entropy for $a \to 0$. Thus, it also follows that $\lim_{a\to 0} \hat{\gamma}_S^{n-1}(X_4) = \lim_{a\to 0} \hat{\gamma}_S^{n-1}(X_3)$.

Now we show that these two last limits have different expressions in terms of the function $\hat{\alpha}_S^n$ when $\theta \to 0$. With the help of the auxiliary sets drawn with dashed and dotted lines in the fig(12)c and (12)d, and applying to these constructions SSB and SSA respectively we obtain

$$\lim_{a\to 0} \hat{\gamma}_S^{n-1}(X_3) \le \hat{\alpha}_S^n(a_1, ..., a_{d-n}) + 2s\, \delta_1^n \mathrm{vol}(R)\,, \tag{68}$$

$$\lim_{a\to 0} \hat{\gamma}_S^{n-1}(X_4) \ge \hat{\alpha}_S^n(2\, a_1, a_2, ..., a_{d-n}) + 2s\, \delta_1^n \mathrm{vol}(R)\,, \tag{69}$$

where we have written explicitly the side lengths of the rectangular polyhedron that enter in the arguments of $\hat{\alpha}_S^n$ and $(a_1, ..., a_{d-n})$ are the side lengths of R. Therefore $\hat{\alpha}_S^n(2\, a_1, a_2, ..., a_{d-n}) = \hat{\alpha}_S^n(a_1, a_2, ..., a_{d-n})$, and from here

$$\hat{\alpha}_S^n(a_1, ..., a_{d-n}) = \hat{\alpha}_S^n(a_1/2^r, ..., a_{d-n})\,, \tag{70}$$

for any integer r. In the limit of large r the right hand side of (70) is just $\hat{\alpha}_S^{n+1}(a_2, ..., a_{d-n})$, which is zero by the induction hypothesis. As the strongly subadditive an positive function $\hat{\alpha}_S^n$ is zero for all rectangular polyhedra it must be zero for all polyhedra, since they can be written as union of simplexes, which in turn can be written as intersections of rectangular polyhedra. This completes the proof of the

Proposition 2: Let S be a d dimensional relativistic entropy function and $\alpha(X) = \gamma(X) - 2s\,\mathrm{vol}(\mathrm{X})$ the $d-1$ dimensional Euclidean function defined in Section IV. If $\alpha_0 = 0$ then $\alpha \equiv 0$.

4. Proof of theorem 3

Finally, we prove the theorem 3. We take S with $\alpha_0 = 0$ and prove $S(X)$ is proportional to the spatial border area of X. From proposition 2 it follows for the flat polyhedra that

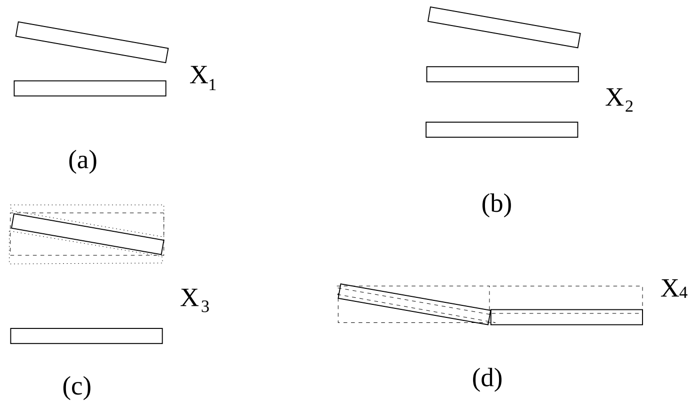

FIG. 12. The entropies of X_1 and X_3 are equal in the limit of small transversal width of their connected components. This can be seen by constructing the auxiliary set X_2, and applying Lemma 2. More generally, we can translate any of the components of X_3 in a direction perpendicular to its principal phase and get a set with the same limit entropy. This shows that X_3 and X_4 have the same limit entropy. The construction in (c) involves a set formed by two parallel rectangular components, the upper component shown with dashed lines and the lower one coinciding with the lower component of X_3, and two wedge sets shown in dotted lines. The application of SSB leads to (68). The construction in (d) involves the set X_4, two wedge sets and a flat rectangular set shown with dashed lines. Application of SSA leads to (69).

$S(X \times [a]) \to 2s\,\mathrm{vol}(X)$ for any $d-1$ dimensional X in the limit $a \to 0$. Thus, in this case the theorem 3 holds, since the boundary area of $X \times [a]$ is $2\,\mathrm{vol}(X)$.

A d dimensional simplex W can be written as $W = \{(x_1, ..., x_d)/(x_1, ..., x_d) = r_1 v_1 + ... + r_{d+1} v_{d+1}$, with $r_i \geq 0$ for $i = 1, ..., d+1$ and $r_1 + ... + r_{d+1} = 1\}$, where $v_1, ..., v_{d+1}$ are the vectors pointing to the simplex vertices. We are interested in a simplex in which $v_{d+1} = u + w$, where the vector u lies inside the simplex phase F with vertices $v_1, ..., v_d$ and w is a vector perpendicular to this phase. We are going to find an upper bound for the entropy $S(W)$ in the limit of small $|w|$. First decompose W as union of the d simplexes formed with vertices u and v_{d+1}, plus any subset of $d-1$ vectors among the vectors $v_1, ..., v_d$. We decompose each of these simplexes as union of wedge sets with small width a, plus a number of small sets, as shown in the fig.(13). In the limit $|w| \to 0$ the previous results show that the entropy of the wedge sets is proportional to the $d-1$ dimensional volume of their big phase. The entropy of the smaller sets is bounded by some constant times the $d-1$ volume of their projection on the plane of the simplex F, since these sets can be formed by intersections of flat wedge and flat rectangular polyhedra.

Taking the limit of small $|w|$ and a we then have

$$\lim_{|w| \to 0} S(W) \leq 2s\,\mathrm{vol}(F)\,. \tag{71}$$

Consider now any simplex U with vertices $u_1, ..., u_{d+1}$ on the hyperplane of time $t = 0$, and let $F_1, ..., F_{d+1}$ be their faces. Call v the centre of the inscribed sphere, the unique sphere that is tangent to all the faces of U, and let r be its radius. The point x with spatial component v at time $t = r$ is at null distance to all $F_1, ..., F_{d+1}$. Call G_i, with $i = 1, ..., d+1$ the null simplex determined by the phase F_i and x, and H_{ij}, with $i \neq j$, to the $d-1$

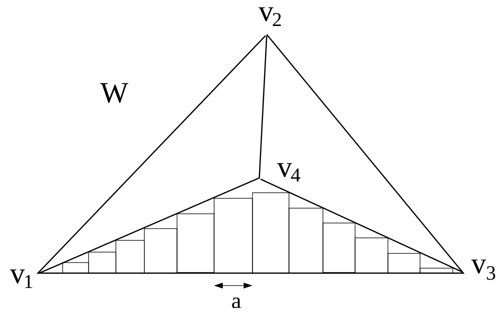

FIG. 13. A three-dimensional flat simplex W with vertices v_1,v_2,v_3 and v_4. The picture shows the projection on the plane of v_1, v_2, v_2. The limit taken in the text is when v_4 approaches a point u on this plane. The decomposition shown for the lower simplex of vertices v_1, v_3, v_4, u is in terms of wedge sets of width a, plus some small sets that can be formed by intersections of wedge and rectangular flat sets. Thus, in the limit $a \to 0$ and $v_4 \to u$ we have the inequality (71) from subadditivity.

dimensional simplex that is the common phase of G_i and G_j (see fig.(14)a). A point y with spatial component v at time $t = r - \epsilon$ would be at small distance from the faces of U for ϵ small. The point y and a phase F_i of U determine a flat simplex G'_i in the above sense, which converge to G_i in the limit $\epsilon \to 0$. The union of the simplexes G'_i for $i = 1, ..., d+1$ form a Cauchy surface for U. We can slightly contract each of the simplexes G'_i in their own hyperplanes to form the simplexes G''_i, which are are at positive distance a from each other (see fig.(14)b). We can further impose to the faces of G''_i and G''_j that are adjacent to each other to be parallel. Then, we add flat elements K_{ij} formed by the Cartesian product of simplexes H'_{ij}, slightly smaller than the H_{ij}, and a small interval $a + \epsilon'$, in such a way to fill in the gaps between the different G''_i and G''_j. We can take all the K_{ij} and the G''_i to be commuting and the intersections of the K_{ij} with G''_i and G''_j to be non empty sets of the form H'_{ij} times small intervals. The union of all the G''_i, the K_{ij}, and a number of sets that are small in more than one perpendicular direction simultaneously and have vanishing small entropy when taking limits, form a Cauchy surface for U. In the limits $y \to x$ and $a \to 0$ the entropy for the flat simplexes G''_i becomes $2s$ times their big phase volume, which coincides with $\mathrm{vol}(F_i)$. The entropy for K_{ij} as well as the one for its intersection with G''_i or G''_j converges to $2s\,\mathrm{vol}(H_{ij})$ in this limit (taking also $H'_{ij} \to H_{ij}$ and $\epsilon' \to 0$). Thus, the application of SSA to this geometrical construction leads to the inequality

$$S(U) \leq 2s \left(\sum_{i=1}^{d+1} \mathrm{vol}(F_i) - \sum_{i=2}^{d+1} \sum_{j=1}^{i-1} \mathrm{vol}(H_{ij}) \right) . \tag{72}$$

Now, as the hypersurfaces G_i are null and have zero expansion, we have that the volume of F_i is equal to the sum of the volumes of the H_{ij} for all j different from i. Thus,

$$\sum_{i=1}^{d+1} \mathrm{vol}(F_i) = 2 \sum_{i=2}^{d+1} \sum_{j=1}^{i-1} \mathrm{vol}(H_{ij}) . \tag{73}$$

From here and the fact that $\sum_{i=1}^{d+1} \mathrm{vol}(F_i) = \mathrm{area}(U)$ by definition, we have

$$S(U) \leq s\, \mathrm{area}(U) . \tag{74}$$

The formula (74) allows us to obtain the corresponding inequality for any relativistic polyhedron X. This can be decomposed into simplexes U_i having empty intersection or intersecting in adjacent faces H_{ij}. We use the same procedure as above. Slightly contract the simplexes U_i and fill the gaps in the Cauchy surface for X with sets K_{ij} formed by product of simplexes H'_{ij} (converging to H_{ij} when taking the limit) and small intervals. The K_{ij} are taken to intersect with the U_i and U_j in sets that are the product of H'_{ij} and small intervals. Then, in the application of SSA to the construction, all the internal faces determined by adjacent simplexes U_i and U_j do not contribute. This is because, according to (74), the contributions from the entropies of U_i and U_j to the term corresponding to the phase H_{ij} add to $2s\,\mathrm{vol}(H_{ij})$, the contribution of K_{ij} converges to $2s\,\mathrm{vol}(H_{ij})$, while on the other side of the inequality, the intersection of K_{ij} with U_i and U_j gives $4s\,\mathrm{vol}(H_{ij})$. Therefore, only the external faces of the U_i (those that are also faces of X) contribute to the inequality, and we have

$$S(X) \leq s\,\mathrm{area}(X) \tag{75}$$

for all X.

Now take a rectangular polyhedron R of sides $(a_1, ..., a_d)$, and call its faces P_i with $i = 1, ..., d$ (R has two faces equal to each P_i). Consider also the sets $P'_i = P_i \times [3a]$ and $P''_i = P_i \times [a, a, a]$, where $[a, a, a]$ is a one dimensional set of two components of size a separated by a distance a. We have $\lim_{a\to 0} S(P'_i) = 2s\,\mathrm{vol}(P_i)$ and $\lim_{a\to 0} S(P''_i) = 4s\,\mathrm{vol}(P_i)$. We can form a big rectangular polyhedron U by union of translated and non intersecting copies of R, the adjacent ones separated by a distance a, and filling the gaps by copies of the P'_i that intersect with the copies of R in sets that are translated copies of P''_i. To complete U we have to add sets that are small in more than one direction. Taking into account (75) the limit of $S(U)/\mathrm{vol}(U)$ for U going to infinity is zero. Thus, the application of SSA to the construction leads in the limit of small a and infinite U to

$$S(R) \geq s\,\mathrm{area}(R)\,. \tag{76}$$

Thus, the equality $S(R) = s\,\mathrm{area}(R)$ holds for any rectangular polyhedron.

Any relativistic polyhedron X is included into some rectangular polyhedron R. Complete a Cauchy surface for X with simplexes to form a Cauchy surface for R. Then, repeating the above procedure for dealing with the internal adjacent boundaries we obtain from SSA

$$S(X) \geq s\,\mathrm{area}(X)\,. \tag{77}$$

This finishes the proof of theorem 3.

VI. DISCUSSION AND CONCLUSIONS

The properties of the quantum entropy make possible to deal with the geometric entropy in a geometrical way. However, even if it is believed that this quantity has a deep physical significance, it remains an abstract concept, wanting for a solid mathematical basis to compute it. In this sense, the principles employed in the paper to obtain the area law are very basic and general, and one can expect them to be still available once a finite geometric entropy is defined.

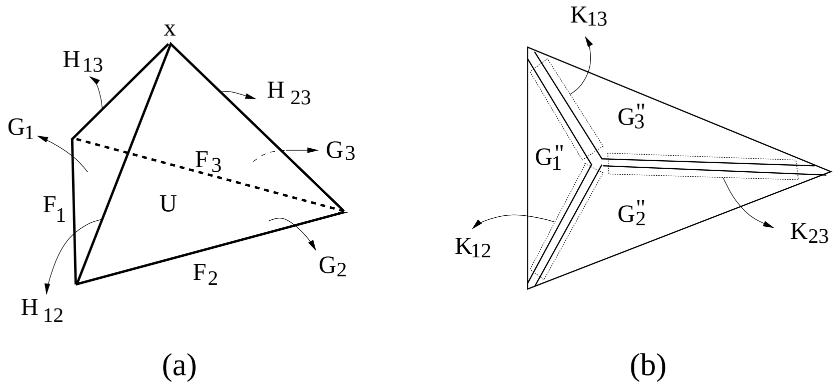

FIG. 14. (a)- The three dimensional set formed by the future domain of dependence of a two dimensional simplex U. The sets F_1, F_2 and F_3 are the faces of U, and the point x is at null distance from these faces. G_1, G_2, and G_3 are the faces of the null boundary of the future domain of dependence, and H_{ij} is the common phase between G_i and G_j. (b)- A view of the set in (a) from the top. The simplexes G_i'' approach the null surfaces G_i and thus they are flat simplexes. With the simplexes G_i'' and the commuting rectangular sets K_{ij} plus some small sets of negligible entropy limit, we form a Cauchy surface for U.

A more immediate meaning can also be given to our results. They indicate that the entropy must be proportional to the area in the limit of infinite cutoff for any regularization scheme in QFT where covariant results are obtained asymptotically. In other words, the leading divergence in the entropy must be proportional to the area for any set. We remark that the theorem shown in this paper tells that no subleading term exists in the continuum, excepting the constant.

This is indeed what all previous QFT calculations have shown for particular cases. Besides, we have also checked the area law for an spherical corona using the numerical method of ref. [5]. This case has also been studied in [19] in a related context. This is a very interesting example since it allows to test a very important and counterintuitive characteristic of the area solution, namely, that it is not ordered by inclusion outside the class of convex sets. As expected, the leading term in the entropy grows with the sum (with equal coefficients) of the areas of the internal and external surfaces of the corona.

There is a curiosity about the Euclidean solutions which could be related both with the uniqueness of the relativistic entropy function and to the divergences. In the Euclidean case it is possible to construct many entropy functions averaging over translations and rotations positive and strongly subadditive functions which are not Euclidean symmetric. Among all these, we can find in particular some very interesting monotonic solutions that are exactly proportional to the area for the convex sets and less than the area for the non convex ones, and where the entropy of two distant objects 'interact' by their mutual shadow. We suspect that a classification of Euclidean solutions could be possible using this averaging method. Coming back to the relativistic case we see that a similar construction does not work since the Lorentz group is non compact, leading to divergent integrals. This is so even taking compact sets to make finite the integration over translations.

An interesting question related to the QFT calculations is the physical meaning of the coefficient relating area and entropy and the constant term. The coefficient of the area term

was related to the Newton constant in [6,23] (see however [8,10,24]), making contact with the black hole entropy. For $1+1$ dimensional conformal field theories it was shown in [9] that the most divergent term for the single interval entropy is a constant (independent of size) proportional to the central charge. This coefficient corresponds to the combination $\gamma = \alpha + \beta$ according to (54). A natural question here is if a new parameter shows up in the expression for the multicomponent set entropy [25]. In other words, the entropy can be simply proportional to the number of components ($\alpha = 0$), or it can contain an additional constant term α, with the same degree of divergence as β. It is possible that this last term is always subleading and negligible in the continuum. This is specially expected in more dimensions since the constant term requires a dimensionless parameter, while the dimensionfull one in front of the area term diverges with the cutoff. Thus, the constant may represent a spurious solution for $d > 1$.

The entropy per unit area s is a function s_ρ of the Poincare symmetric state ρ. The combination of the inequalities (2) and Theorem 3 imply that given any two such states we have

$$s_{\lambda_1\rho_1+\lambda_2\rho_2} = \lambda_1 s_{\rho_1} + \lambda_2 s_{\rho_2}\,, \tag{78}$$

for positive λ_1 and λ_2 with $\lambda_1 + \lambda_2 = 1$. Thus the entropy per unit area is afin under mixing of states, a property known for the mean entropy per unit volume of traslational invariant states. The physical meaning of eq. (78) is however unclear to the author.

In a series of recent papers a very interesting phenomena has been pointed out [11,26]. The authors discovered that the entropy in a volume is not the only quantity that scales with the area of the given region but the same behavior is expected for a whole class of operators averaged using the local density matrices. In particular they showed the area law for the energy fluctuations. The results of this paper rely heavily on the use of the strong subadditive property of the entropy, and thus they can not be extended directly to operator correlations. However, it is possible that the method here could be generalized by introducing chemical potentials for suitable operators attached to the local regions.

The proof in this paper uses explicitly arbitrarily small and large distances. The infiniteness of the space is specially relevant in the argument around the constructions of fig.(12), which involve large translations, and lead to the equation $\gamma(X) = \beta(X)$. This is what excludes the possibility of finding a solution ordered by inclusion. However, it is not excluded that different type of solutions exist in de Sitter space. This has compact Cauchy surfaces, and it is also maximally symmetric, what most probably can allow us to make a complete classification of solutions as the one given here for the Minkowski space [25].

The entropy of the black hole and the geometric entropy are 'empty' space quantities. There are a number of proposals for general bounds to the entropy of bounded systems which are suggested by the physics of black holes. They have been successfully checked mainly in the classical regime. The definition of a bounded system requires a partition of the Hilbert space into a tensor product, and it is necessary to add boundary conditions. Thus, at the semiclassical level, the entropy bounds should include a term coming from the vacuum correlations, even in empty space (see similar ideas in [20]). In principle, this could greatly tighten the inequalities. How do these bounds compare with our result in flat space?. Consider the covariant bound proposed in [27], which is related both with the Bekenstein bound [28] and the generalized second law. Take a $d-1$ dimensional spatial surface Ω_1

and a null congruence $\mathcal{W}$ orthogonal to it, having non positive expansion, and ending in another $d-1$ dimensional spatial surface Ω_2. The bound states that the entropy crossing $\mathcal{W}$ is bounded above by a constant times $\mathrm{vol}(\Omega_1) - \mathrm{vol}(\Omega_2)$. Thus, this gives zero entropy for our polyhedra, since they have Cauchy surfaces formed by union of null flat surfaces having zero expansion.

This touches upon the question of the geometric entropy for a more general type of subsets than the one considered here. The area and the constant term are not the most general solutions of the SSA inequalities for larger classes of subsets than the polyhedra. Take for example the class of subsets with piecewise differentiable spatial corner. It is also a closed class under intersection and union of commuting sets. As can be easily seen from fig(1)b, the integral over the spatial corner of any geometrical quantity that is positive and local gives place to a relativistic entropy function. Of course, the theorem 3 implies that these solutions are proportional to the area on the polyhedra. As an example of such a local quantity we can take the square of the expansion of the future null boundary of the given causally closed set. In this case the entropy is non zero for the sphere and zero for every polyhedron approaching it. This shows that this type of solutions has a very pathological discontinuous behavior.

Excluding these solutions, probably unphysical, it seems we are left with two possibilities, either the covariant bound is wrong outside the classical domain, or the assumptions of this paper fail somewhere. However, it is also possible and likely that the covariant bound has to be applied to a regularized notion of entropy which is zero on Minkowski space. This is appealing, since the calculation of the black hole entropy using the Euclidean path integral requires the subtraction of the corresponding flat space quantity [29].

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